Lineare Algebra Zusammenfassung

Normalice a vector:

Devide every component by the $\underbrace{\text{length of the vector}}_{\text{denoted as } |v| \text{ or } \|v\|}$

Perpendicular:

two vectors are perpendicular if the dot product is 0

angle between vectors

$$\theta = \arccos \frac{\vec{v} \cdot \vec{u}}{|v|\,|u|}$$

Vector Multiplication:

$$\vec{v} \cdot \vec{u} = \begin{pmatrix} v_1 \cdot u_1 \\ v_2 \cdot u_2 \\ \vdots \quad \vdots \\ v_n \cdot u_n \end{pmatrix}$$

Unit vector

vector is a unit vector if length $= 1$

Matrices

$m \times n$ Matrix

no. rows — no columns

row echelon form

Leading 1's, no numbers below $\begin{pmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$

row reduced form

row echelon form but also no numbers above. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Matrix Multiplication:

$$v \cdot u = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix}$$

$$= \left(\begin{array}{c|c} v_{11} \cdot u_{11} + v_{12} \cdot u_{21} + v_{13} \cdot u_{31} & v_{11} \cdot u_{12} + v_{12} \cdot u_{22} + v_{13} \cdot u_{32} \\ v_{21} \cdot u_{11} + v_{22} \cdot u_{21} + v_{23} \cdot u_{31} & v_{21} \cdot u_{12} + v_{22} \cdot u_{22} + v_{23} \cdot u_{32} \\ v_{31} \cdot u_{11} + v_{32} \cdot u_{21} + v_{33} \cdot u_{31} & v_{31} \cdot u_{12} + v_{32} \cdot u_{22} + v_{33} \cdot u_{32} \end{array}\right)$$

number of columns of v has to match the number of rows of u

$$\begin{array}{ccc} & & \\ \end{array}\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b \\ \\ \end{pmatrix} \Rightarrow \text{see as} \begin{array}{l} x_1 \cdot \text{column } 1 \\ x_2 \cdot \text{column } 2 \\ x_3 \cdot \text{column } 3 \end{array}$$

col 1 — col 2 — col 3

Inverse

A Matrix is invertible if Determinant is NOT 0

Solving linear equation with Inverse

$$Ax = b \Rightarrow x = A^{-1} \cdot b$$

Elementary Matrix

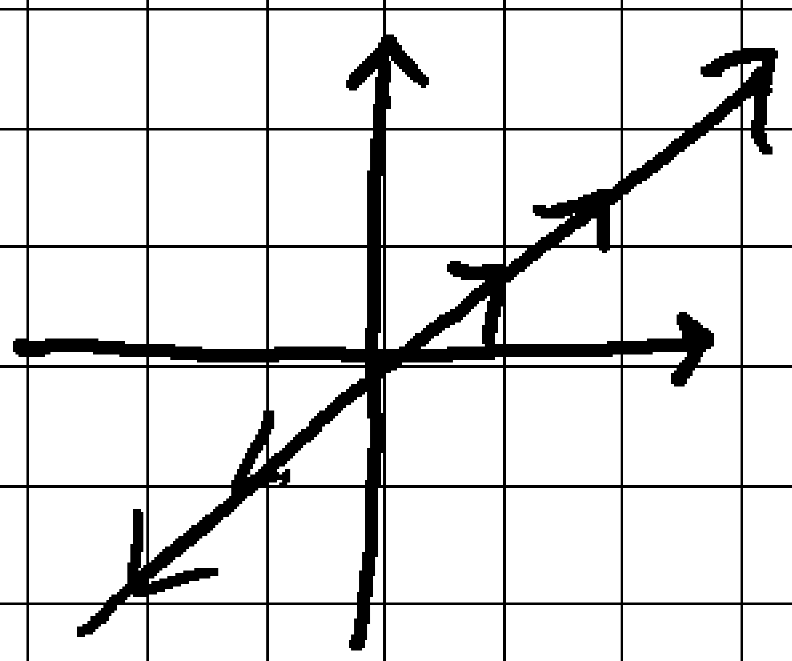
This a matrix of a single elementary row operation starting from the Identity Matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = r_2 \cdot 2$

Product of elementary Matrices

multiple elementary row operations $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{matrix} = r_1 + r_2 \\ = r_2 \cdot 2 \end{matrix}$

Subspace

Subspace $\rightarrow$ vector space inside another vector space



$\rightarrow$ Line in $\mathbb{R}^2$ through zero vector

every subspace has to contain the zero vector because you have to be allowed to multiply by 0.

Subspaces of $\mathbb{R}^2$

1. all of $\mathbb{R}^2$
2. any line through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
3. zero vector only (Z)

Subspaces of $\mathbb{R}^3$

1. all of $\mathbb{R}^2$
2. a plane through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
3. a line through $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
4. zero vector only

Column Space

The column space contains all linear combinations of the columns.

Nullspace ($N(A)$) also called kernel

The Nullspace contains all combination of all solutions of $Ax = 0$. The number of solutions is equal to the number of free variables of the system of equations.
The number of free variables is the number of unknowns, the number of columns, minus the rank.

$\Rightarrow n - r$ = no. of free variables = nullity of Matrix

Nullity of a Matrix

$$\underbrace{n}_{\text{no. of columns of } A} - \underbrace{r}_{\text{rank of } A} = \text{nullity of } A$$

Rank of a Matrix

The Rank of a Matrix is the number of linearly independent rows or columns in the matrix
(number of pivots)

Independence / Dependence

Vectors are Independent if no combination gives the zero vector (exept the zero combination)

$c_1 x_1 + c_2 x_2 + \dots + c_n x_n \neq 0$ (exept if all c's are 0)

↳ if one of the vectors is the zero vector they are always dependent, because $n \cdot$ zero vector remains the zero-vector.

They are independent if nullspace of A is $\begin{Bmatrix} \text{zero} \\ \text{vector} \end{Bmatrix}$

↳ rank $= n$ (no free variables)

They are dependent if $Ac = 0$ for any non-zero c

↳ rank $< n$ (there are free variables)

Spanning a space

Vectors span a Space means:

The space consists of all combination of those vectors.

Basis of vector space

Basis for a vector space is a sequence of vectors with 2 properties.

1. They are independent
2. They span the space

Given a space, every basis for said space has the same no. of vectors.

this is the dimension of the space

Dimension of a space

The number of vectors of a basis for a space is called the dimension of the space.

Four fundamental subspaces

① column Space $C(A)$

② Null space $N(A)$

③ Row space all combination of rows of A

= all combination of columns of A^T

= $C(A^T)$

④ Null space of A^T = $N(A^T)$

↳ left Null space

orthogonal vectors

2 vectors are orthogonal, if the dot product is 0

Two subspaces of a given space are orthogonal, if all the vectors of the two subspaces are perpendicular to each other.

If two subspaces intersect in any non-zero-vector the two subspaces are not orthogonal

the row space is orthogonal to the nullspace

the column space is orthagonal to the left nullspace $(N(A^T))$

orthagonal subspaces are each others orthogonal complement

Projection

If $Ax = b$ has no solution, we find the solution with the smallest possible error

$\Rightarrow$ we multiply both sides with A^T

$$\Rightarrow \quad A^T A \hat{x} = A^T b$$

$$\Rightarrow \hat{x} = (A^T A)^{-1} A^T b$$

Projection Matrix P

$$P = A \hat{x} = A (A^T A)^{-1} A^T$$

P is symetric $\Rightarrow$ $\boxed{\begin{array}{l} P = P^T \\ P^2 = P \end{array}}$

also

Proj. Matrix $= \underline{A(A^T A)^{-1} A^T}$

$\tilde{p}$ projection $= Pb = A(A^T A)^{-1} A^T \cdot b$

$\Downarrow$

if b in the column space, then $Pb = b$

$\hookrightarrow Pb = A(A^T A)^{-1} A^T \cdot \underbrace{b}_{Ax} = A\underbrace{(A^T A)^{-1} A^T A}_{=1} x = Ax = b$

if b is $\perp$ to the column space, then $Pb = 0$

$\hookrightarrow$ b is in the nullspace so Pb is automatically 0

Determinants

2x2 Matrix : $a_{11} \cdot a_{22} - a_{12} a_{21}$

3x3 Matrix : $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

3 rechts diagonalen addiert
−
3 links diagonalen subtrahiert

Cofactor expansion

· optimale Reihe auswählen (möglichst viel 0en)

$$\boxed{C_{ij} = M_{ij}}$$

$$\det(A) = a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \ldots + a_{in} \cdot C_{in}$$

↳ $C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$

Reihe (i), Spalte (j)

↳ A_{ij} = Matrix ohne Zeile i und ohne Spalte j

Adjugate Matrix
→ Matrix of cofactors $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ \vdots & \end{pmatrix}$

Cramers Rule

$$A \quad x = b$$

$x_1 = \frac{\det A}{\det A_{x_1}}$ ⇒ A_{x_1} = A aber erste Spalke (x-wert) ersetzt durch b

$x_2 = \frac{\det A}{\det A_{x_2}}$ ⇒ A_{x_2} = A ohne Spalte 2 (y-werte) ersetzt durch b

$x_3 = \frac{\det A}{\det A_{x_3}}$ ⇒ A_{x_3} = A ohne Spalte 3 (z-werte) ersetzt durch b

z.B $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \Rightarrow A_{x_1} = \begin{pmatrix} 4 & 2 & 3 \\ 5 & 5 & 6 \\ 6 & 8 & 9 \end{pmatrix}$

Determinant by Gauss elimination

- Compute row reduced form
  ↳ on row exchange $\det(A) = -\det(A)$
- multiply elements of main diagonal $= \det(A)$

Eigenvalues

existiert nur für quadrat Matrizen

Eigenvalues λ_n of A =

Hauptdiagonale einer Matrix alle Elemente $-\lambda$

↳ $\underbrace{\text{Determinante} \stackrel{!}{=} 0}$ $\Rightarrow$ λ berechnen

Det berechnung mit det = 0 und λ eingesetzt

↳ Tipp: häufig is Element der Hauptdiagonale oder 0 eine Lösung

$\Rightarrow$ Kontrolle → Summe der Hauptdiagonale = Summe der Eigenvalues

Eigenvectors

EV zu λ_1 $\Rightarrow$ Gleichungssystem lösen von

$A \rightarrow$ Hauptdiagonale $- \lambda_1$ $\begin{pmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_1 \end{pmatrix}$

$\Rightarrow$ row reducen

↳ Parameter für 0 Reihe definieren

↳ auflösen

↳ Parameter ausklammern = EV

Diagonalisierung

↳ Transformation zu Dreiecksmatrix

$S^{-1}AS = D \quad , \quad A = SDS^{-1}$

D = Matrix mit der Hauptdiagonale = den Eigenvalues von A

S = Matrix der Eigenvectors von A (Reihenfolge gleich wie bei D)

Similarity

2 Matrices are similar if det(A) = det(B)

Linear Transformations

Transformation Matrix

Transforms vectors from one Basis to another

↳ perform same row operation on both Matrices bring 1 to I
→ resulting Matrix is the Transformation Matrix